The Statistical Prediction of Voltage Standing-Wave Ratio*

J. A. MULLEN† AND W. L. PRITCHARD‡

Summary—The problem of predicting the probability distribution of vswr for many randomly spaced discontinuities is solved using the "central limit theorem." Assuming that reflection factors add in the complex plane and using the "central limit theorem" the result is shown to be a Rayleigh distribution in terms of the reflection factor.

The probability of the vswr over a band of frequencies is calculated using the concept that this band of frequencies can be considered as a number of statistically independent samples.

Distribution Function for Voltage Standing-Wave Ratio

The designer of microwave systems is faced with the problem of estimating the voltage standing-wave ratio resulting from the combined reflections of many components. Typically, a radar system may contain an antenna reflector, radome, horn feed, polarizing devices, two or three rotating joints, a waveguide switch, duplexer, and perhaps twenty or thirty bends, twists, and flanges. In the worst conceivable case the over-all voltage standing-wave ratio will be the product of the component voltage standing-wave ratios. With reasonably attainable values of voltage standing-wave ratio for the components this worst case is often too horrible to contemplate. Conversely, an acceptable value of maximum over-all voltage standing-wave ratio leads to impossibly small values for the components. An exact computation of the over-all voltage standing-wave ratio from the values of component voltage standing-wave ratios and line lengths is extraordinarily complicated. Even worse, the line lengths are often not known until late in the system development and long after the components are designed. In the past, equipment designers have relied on accumulated experience to estimate the practical result, *i.e.*, to take advantage of the small statistical probability that the individual mismatches will combine to give the worst case.

It is the purpose of this paper to treat the problem statistically, *i.e.*, to obtain the distribution function of over-all standing-wave ratio in terms of the number of discontinuities and their mean squared reflection factor.

Note that we are not considering the production problem of quality control which is that of predicting the variation of voltage standing-wave ratio of an assembly when the line lengths vary from their design values. This problem has been dealt with by Brown.[1]

In order to make the problem analytically tractable we assume that the reflection factors (complex) add, *i.e.*,

$$\hat{\gamma} = \sum_{i=1}^{n} \gamma_i \epsilon^{j2\theta_i}$$

$$\hat{\gamma} = \sum_{i=1}^{n} \gamma_i \cos 2\theta_i + j \sum_{i=1}^{n} \gamma_i \sin 2\theta_i$$

$$\hat{\gamma} = \gamma_x + j\gamma_y. \tag{1}$$

This is a reasonable approximation if the magnitude of the individual reflection factors are small. Practical system designs usually involve magnitudes of γ well within these limits. It should be noted that, mathematically at least, this assumption permits over-all magnitudes of γ greater than unity, which is physically impossible. However, the mathematics will also show that the probability of γ anywhere near unity (where the additive approximation is invalid) is exceedingly slight.

We further assume that the line lengths between discontinuities are independent of each other and that all values of θ between 0 and 2π are equally probable. These assumptions seem reasonable in view of the fact that electrical line lengths in a complicated microwave system are generally much longer than 2π and are chosen for random mechanical reasons.

We finally assume that n, the total number of discontinuities, is large ($n > 8$). If the system has periodic discontinuities, *e.g.*, flanges recurring at equal intervals, they should be considered as a single discontinuity, the magnitude of which is calculable separately and exactly.

These assumptions are sufficient to use the "central limit theorem" of probability theory.[2]

The central limit theorem can be stated for our purposes as follows: the joint distribution function of the real and imaginary parts of the sum of a large number of independent complex random variables asymptotically approaches normal regardless of the distribution functions of the individual random variables.[3]

Eq. (1) for the complex reflection factor is the sum of a large number of independent random variables

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† Raytheon Mfg. Co., Waltham, Mass.
‡ Raytheon Mfg. Co., Wayland, Mass.

[1] L. W. Brown, "Problems and practice in the production of waveguide transmission systems," *Proc. IEE*, vol. 93A, pp. 639–646; 1946.

[2] J. V. Uspensky, "Introduction to Mathematical Probability," McGraw-Hill Book Co., New York, N. Y. ch. 15; 1947.

[3] There are only slight restrictions not applicable here.

and, as such, comes within the scope of the central limit theorem.

The joint distribution of the real and imaginary components of $\hat{\gamma}$ is now given in the two dimensional normal form.[4]

$$W(\gamma_x, \gamma_y) = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \exp\left\{-\frac{\gamma_x^2 - 2r\gamma_x\gamma_y + \gamma_y^2}{2\sigma^2(1-r^2)}\right\}. \quad (2)$$

Eq. (2) is the joint normal distribution where the mean values of γ_x and γ_y vanish and where the mean squared values are equal and calculated as follows:

$$\gamma_x^2 = \sum_i \sum_j \gamma_i \cos 2\theta_i \gamma_j \cos 2\theta_j. \quad (3)$$

In averaging the right hand side over θ_i and θ_j their independence when $i \neq j$ causes their joint average to be the same as their separate averages, and the uniform distribution of θ_i and θ_j causes these averages to be zero. Thus we can write

$$\overline{\gamma_x^2} = \overline{\sum_i \gamma_i^2 \cos^2 2\theta_i} \quad (4)$$

$$= \sum_i \gamma_i^2 \overline{\cos^2 2\theta_i}. \quad (4a)$$

Since θ_i is uniformly distributed we can make use of the fact that the mean squared value of a cosine is equal to $\frac{1}{2}$ and write:

$$\overline{\gamma_x^2} = 1/2 \sum_{i=1}^{n} \gamma_i^2. \quad (5)$$

Similarly

$$\overline{\gamma_y^2} = 1/2 \sum_{i=1}^{n} \gamma_i^2$$

and by definition

$$\sigma^2 = \overline{\gamma_x^2} = \overline{\gamma_y^2}. \quad (5a)$$

We must now calculate r, the correlation coefficient, which is defined as follows:

$$\sigma^2 r = \overline{\gamma_x \gamma_y}. \quad (6)$$

Using an argument identical to that following (3) we proceed:

$$\sigma^2 r = \overline{\sum_i \sum_j \gamma_i \cos 2\theta_i \gamma_j \sin 2\theta_j \delta_{ij}}$$

$$= \sum_i \gamma_i^2 \overline{\cos 2\theta_i \sin 2\theta_i}$$

$$= 1/2 \sum_i \overline{\gamma_i^2 \sin 4\theta_j}.$$

Thus

$$\sigma r = 0. \quad (6a)$$

Now, we can write the joint distribution simplified as:

$$W(\gamma_x, \gamma_y) = \frac{1}{2\pi\sigma^2} \exp -\left(\frac{\gamma_x^2 + \gamma_y^2}{2\sigma^2}\right). \quad (7)$$

Since we are seeking the distribution of the magnitude of the reflection factor we must transform from rectangular to polar coordinates.

The element of area goes from $d\gamma_x d\gamma_y$ to $\gamma d\gamma d\theta$. With γ_x and γ_y expressed in polar coordinates, the joint density function becomes

$$W(\gamma_x, \gamma_y) d\gamma_x d\gamma_y = \frac{1}{2\pi\sigma^2} \epsilon^{-\gamma^2/2\sigma^2} \gamma d\gamma d\theta. \quad (8)$$

From the definition of $W(\gamma, \theta)$[5] we have that

$$W(\gamma, \theta) = \frac{\gamma}{2\pi\sigma^2} \epsilon^{-\gamma^2/2\sigma^2}. \quad (9)$$

By integrating over θ the distribution of γ is seen to be

$$W(\gamma) = \frac{\gamma}{\sigma^2} \epsilon^{-\gamma^2/2\sigma^2}. \quad (10)$$

This is the well-known Rayleigh distribution. In fact we have just rederived in a different context the two dimensional "random walk" problem for a large number of steps.[6–8]

The value which maximizes $W(\gamma)$ is defined as the most probable value of γ and is designated γ_m. Since γ_m is found equal to σ, we have

$$W(\gamma) = \frac{\gamma}{\gamma_m^2} \epsilon^{-\gamma^2/2\gamma_m^2}. \quad (11)$$

The distribution of over-all reflection factor has been plotted using (11) for representative values of γ_m in Fig. 1. Note that the smaller γ_m, the sharper and higher is the distribution function. The most probable value, γ_m, is calculated from the individual reflection factors as follows:

$$\gamma_m = \sigma = \frac{1}{\sqrt{2}}\left(\sum_{i=1}^{n} \gamma_i^2\right)^{1/2}. \quad (12)$$

If γ_0 is the rms value of the γ_i's, we can rewrite (12) to show explicitly the dependence on n as

$$\gamma_m = \sqrt{\frac{n}{2}}\, \gamma_0. \quad (13)$$

[4] H. Cramer, "The Elements of Probability Theory," John Wiley and Sons, New York, N. Y., ch. 9, sec. 4; 1955.

[5] $W(\gamma, \theta)$ stands for the probability density of the random variable within the parentheses and does *not* reprersent the same function for different random variables.

[6] L. Rayleigh, *Phil. Mag.*, vol. 10, p. 73; 1880.

[7] K. Pearson, Drapers Co. Res. Memo No. 4, Biometric Series No. III; 1906.

[8] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," Rad. Lab. Ser. No. 24, McGraw-Hill Book Co., Inc., New York, N. Y.; 1950.

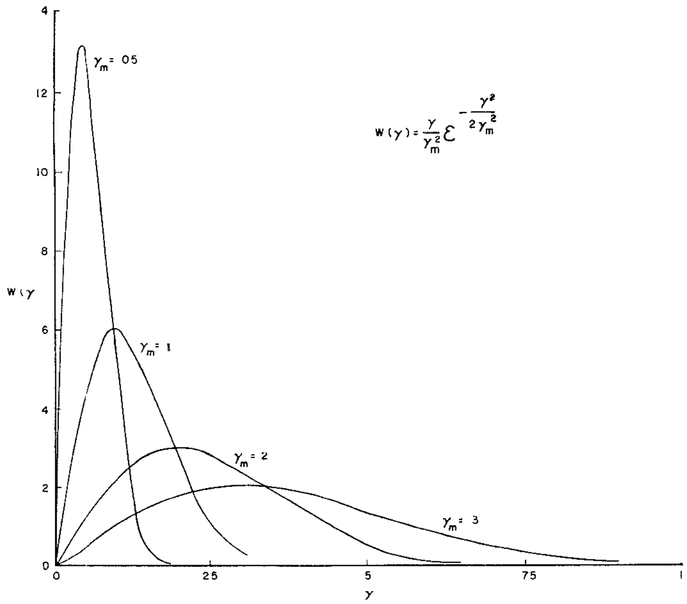

Fig. 1—Probability density of reflection factor with the most probable over-all reflection factor as parameter.

We note that the probability density in the region where the additive approximation becomes questionable ($\gamma \cong 0.5$) is extremely small.

The probability $P(\gamma)$ that the reflection factor is less than γ is given by

$$P(\gamma) = \int_0^\gamma W(\gamma)d\gamma = 1 - \epsilon^{-\gamma^2/2\gamma_m^2}. \tag{14}$$

The convenience of the results is increased by writing the probability in terms of ρ, the vswr, rather than γ. γ and ρ are related by

$$\gamma = \frac{\rho - 1}{\rho + 1} \cdot \tag{15}$$

Defining ρ_m to correspond to γ_m using (15), we can write

$$P(\rho) = 1 - \exp\left\{-1/2\left(\frac{\rho - 1}{\rho + 1}\right)^2\left(\frac{\rho_m + 1}{\rho_m - 1}\right)^2\right\}. \tag{16}$$

$P(\rho)$ is plotted in Fig. 2 for the same values of ρ_m as in Fig. 1. These curves can be used to compute the probability that, among a large number of possible designs with the same set of discontinuities, the vswr of a particular design will be less than ρ. Note that the probability of being less than ρ_m is only 0.4, so that ρ_m is not a conclusive design parameter, but for moderately larger values of ρ the asymptotic approach to unity is rapid.

To plot these results in a still more useful form we define $\rho_{.9}$ as that value of vswr that we have only a ten per cent probability of exceeding. From (14) we find

$$\gamma_{.9} = \sqrt{2 \operatorname{Log} 10}\,\gamma_m = 1.52\sqrt{n}\,\gamma_0 \tag{17}$$

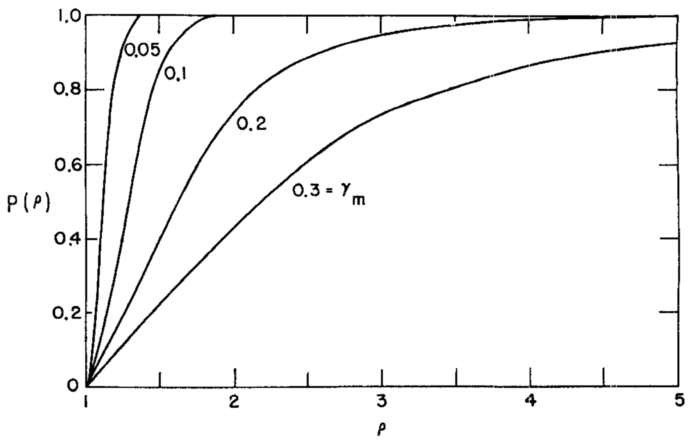

Fig. 2—Cumulative probability of voltage standing-wave ratio with the most probable over-all reflection factor as parameter.

whence

$$\rho_{.9} = \frac{1 + \gamma_{.9}}{1 - \gamma_{.9}} \cdot \tag{17a}$$

Eq. (17), using the variable of (17a), is plotted in Fig. 3 for representative values of n. These curves permit a system designer to predict, given a number of discontinuities and their typical values, a very conservative result for the over-all vswr. Using the preceding methods a set of curves is easily constructed to predict a less conservative result, *e.g.*, $P(\rho) = \frac{2}{3}$.

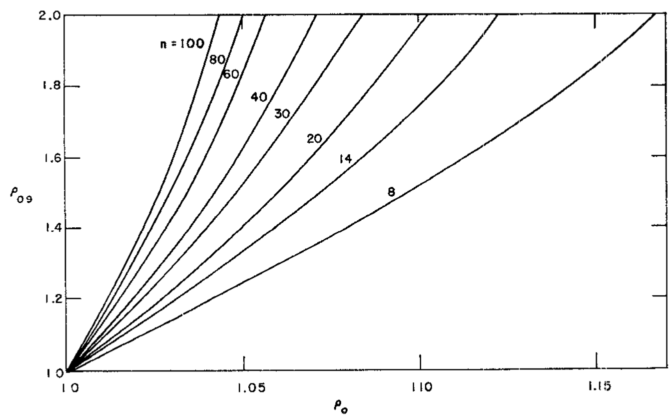

Fig. 3—The value of vswer exceeded in only 10 per cent of the possible designs plotted vs the vswr of a typical discontinuity, with the number of discontinuities as a parameter.

Prediction Over a Wide-Frequency Band

So far the theory has considered results at a single frequency only and cannot be used to predict results over a band of frequencies. We have approached this latter problem by considering it as a "sampling" problem. In other words, if the frequency range is wide, we actually have a multiplicity of problems, each the same as that solved in the first section. The difficulty, of course, is in deciding how many independent samples the frequency band contains. We estimate this number and show that the answer is insensitive to this estimate.

There are two possibilities, that the reflection factor is greater than γ or less than γ. The joint probability that in N tries, represented by the N independent frequency points, there will be N favorable results, *i.e.*, vswr less than γ, is simply $P^N(\gamma)$.[9]

$P^N(\gamma)$ is plotted in Fig. 4 using a normalized abscissa scale. Note that for a constant probability the value of γ varies about as log N. This dependence is insensitive to N which is fortunate since the awkward aspect of this problem is in deciding on the number of independent frequencies.

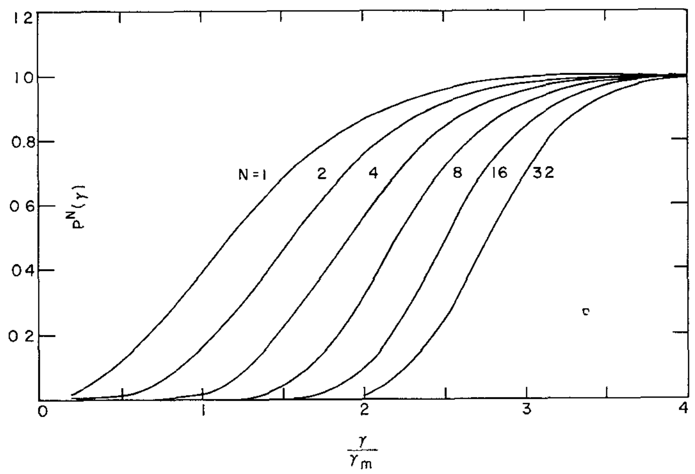

Fig. 4—The cumulative probability that the reflection factor is less than γ in a bandwidth B plotted against γ/γ_m with the number of independent samples (determined from B) as parameter.

Our method is based on the following considerations. When the frequency has changed sufficiently that the average electrical length between discontinuities has changed by π, the individual reflection factors have changed their phases enough so that their sum can be considered as a new random variable. On this basis, N is given by

$$N = \frac{B}{\Delta f} = 2\,\frac{B}{f}\,\frac{L}{\lambda}\,. \tag{18}$$

This estimate of N is optimistic and realistic system planning should use a value of N probably twice that given by (18).

In many typical systems, one length (*e.g.*, a smooth run from transmitter to antenna) is much longer than any of the others. This problem should not be considered statistically on an over-all basis, but should be resolved into two or more separate statistical problems, whose results are combined by conventional methods. In other words, at some point in the frequency band, corresponding to a single independent sample, two groups of discontinuities separated by a very long length of line will combine in the most unfavorable phase; so that the most probable vswr is the product of the most probable vswr's of the separate groups.

If the magnitude of the vswr's of the individual discontinuities varies over the band, a conservative approach to system design would use the largest values in calculating γ_0.

[9] We have used γ rather than ρ as an independent variable in order to achieve a universal curve.

Conclusion

We have determined the probability distribution at a single frequency for many randomly spaced small discontinuities, and plotted results in several convenient forms. This theory has been extended in an intuitive fashion to cover a band of frequencies.

List of Symbols

- γ magnitude of over-all reflection factor.
- $\hat{\gamma}$ over-all complex reflection factor.
- γ_i magnitude of the reflection factor of the ith discontinuity.
- γ_x real component of over-all reflection factor.
- γ_y imaginary component of over-all reflection factor.
- γ_0 rms value of the γ_i's.
- γ_m the most probable value of γ.
- λ midband wavelength.
- θ_i electrical line length, in excess of a whole number of 2π's, to ith discontinuity.
- n total number of discontinuities.
- r correlation coefficient between γ_x and γ_y.
- σ standard deviation of either γ_x or γ_y.
- δ_{ij} Kronecker delta $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.
- L average length between discontinuities.
- $W(\cdot)$ probability *density* (distribution function) of variable included in parentheses.
- N number of independent samples in the system bandwidth.
- ρ over-all voltage standing-wave ratio.
- $\rho_{.9}$ that value of over-all vswr exceeded in only 10 per cent of the possible designs.
- ρ_m the most probable value of ρ.
- $P(\cdot)$ the probability that the random variable is less than the value in parentheses.
- B total system bandwidth.
- f midband frequency.
- — a bar over any term indicates the taking of its arithmetic mean.
- Δf frequency spacing between independent samples.